24

SOME STUDIES ON THE EFFECTS OF LIMITING IN A POSITION CONTROL SERVO CONTAINING BACKLASH

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ABSTRACT. Effects of limiting on the stability of a second order position control servomechanism containing backlash in the output coupling have been studied. Two cases have been considered, the signal being limited is, in the first case, θ_m, the motor speed, and in the second case $\theta_m + a\theta_m$, a being a constant. Describing function method was applied for theoretical analyses and experimental studies were made with the help of a simulator.

1 INTRODUCTION

All servo systems are in practice nonlinear—the nonlinearities arising from limitations of the different components. The most common and important froms of nonlinearities encountered arise from (i) saturation or limiting of the response, (ii) dead-zone, (iii) backlash in different linkages and (iv) nonlinear load frictions, such as, static and coulomb frictions. The individual effects of these nonlinearities on the performance of a servo system have been fairly widely studied. Results indicate that from the point of view of system stability backlash may be one of the most disquieting factors.

Tustin (1947) and Liversidge (1952) have made detailed studies of systems containing backlash in output couplings and have suggested means for countering the adverse effects of backlash on system stability. In these studies presence of other nonlinearities were not considered, i.e. the entire system was assumed to behave linearly except for the backlash element. As is well known such ideal conditions seldom occur in practice. It is therefore important to study the effects of simultaneous occurrence of more than one nonlinearity in the system. The present paper describes results of some preliminary studies made on the performance of a servomechanism effected by limiting and backlash in the output coupling, with particular reference to the question of stability. A second order position control system has been considered.

There is no readily applicable method of analysis suitable for systems with multiple nonlinearities. A modification of the describing function method (Johnson, 1952) has here been utilised for theoretical analysis aimed at qualitative results. The experimental part of the analysis involved studies with the help of an analog simulator.

2 THE BASIC SYSTEM

The basic system under consideration is shown in figure 1 in the form of a block diagram. The output of the error detector is amplified by an amplifier of gain K'. The amplifier output is the control signal of the motor. The motor is assumed

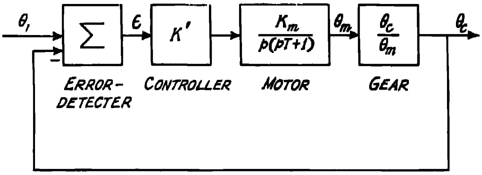

Fig. 1. The basic system under consideration.

to be adequately representable by two time constants as indicated in figure 1, viz., T and ∞, the latter being the integrating one. The motor shaft is coupled to the load shaft by means of a backlash element; θ_m and θ_c are respectively the motor and load positions. The backlash characteristic assumed is shown in figure 2. The load inertia is neglected and it is also assumed that the load is sufficiently damped so as to prevent any overcoasting during reversals.

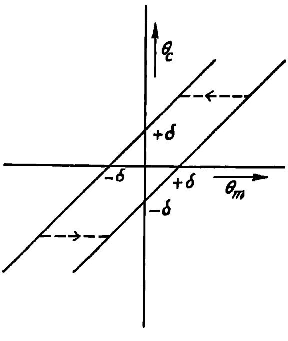

Fig. 2. Assumed blacklash characteristic of the coupling gear.

This system was studied in some details by Nichols (1953). It has been shown that the effect of backlash on system stability depends on the value of the amplifier gain K'. For small values of K' backlash merely deteriorates the relative stability of the system. If K' is larger than a certain critical value backlash produces absolute instability and causes the system to sustain oscillations of definite amplitude and frequency. We shall here extend the study of Nichols to examine the effects of limiting which may be inherently present or may be introduced intentionally in the system of figure 1.

For convenience we shall assume that the motor time constants are separable. This assumption may appear to be somewhat impracticable but will be shown later to be justified on analytical grounds. Two different cases studied are shown in figures 3(a) and 3(b). In case of figure 3(a) it is easily verified that the signal

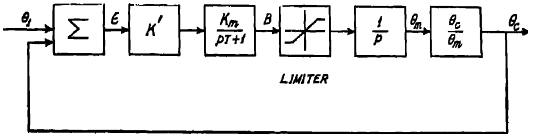

Fig. 3(a). Schematic diagram of the system with simple velocity limiting.

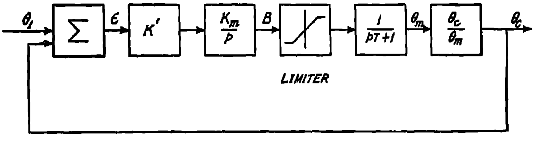

Fig 3(b). Schematic diagram of the system with modified velocity limiting

which is limited is $\dot{\theta}_m$, whereas in the case of figure 3(b) the corresponding signal is $\dot{\theta}_m + \alpha\theta_m$ where $\dot{\theta}_m = \frac{d\theta_m}{dt}$ and α is a constant. We shall refer to the case of figure 3(a) as simple velocity limiting and to that of figure 3(b) as modified velocity limiting.

3. SIMPLE VELOCITY LIMITING

The system is shown in figure 3(a). The limiter characteristic assumed is shown in figure 4. The describing function of the limiter is given by (Johnson, 1952)

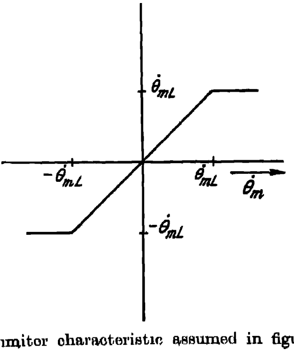

Fig. 4. Limiter characteristic assumed in figure 3(a).

If we consider a point Q on the $-G_B$ locus corresponding to a definite input signal amplitude, it is possible to find a point P on the G^{-1} locus corresponding to the same signal amplitude, which is nearest to Q. Then

$$\frac{OQ}{QP} = \frac{G_B}{[G^{-1}+G_B]_{min.}} = \left(\frac{\theta_c}{\theta_1}\right)_P$$

where, from relation (6) $(\theta_c/\theta_1)_P$ is the maximum value of the ratio (θ_c/θ_1), which is analogous to the closed-loop frequency response peak (M_P) (Brown, 1955) of the linear theory. The value of $(\theta_c/\theta_1)_P$ gives an estimate of the damping present in the system, larger value corresponding to lesser damping and vice versa. Also the frequency ω_P corresponding to P gives an idea of the speed of response, larger values of ω_P corresponding to greater speed of response.

It can be seen from figure 6 that if we move on to another point Q' corresponding to a signal amplitude larger than that at Q, the magnitude of this ratio decreases. That is, the system damping increases as the signal amplitude increases Also the frequency ω_P decreases with signal amplitude and so the system becomes more sluggish.

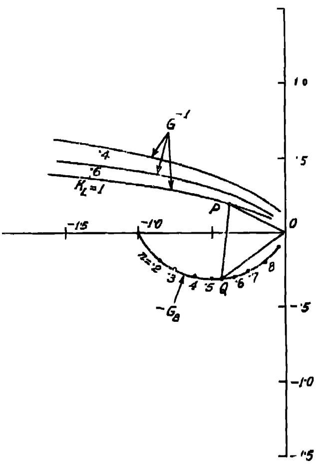

Fig. 6. Complex plots of G^{-1} and $-G_B$ for the system in figure 5(a) with $K = 10$ and $T = 1$ sec.

Records of responses to step inputs of the system in figure 3(a) obtained with the help of the simulator are shown in figure 7. Here for convenience of comparison, the magnitude of the input step was kept constant and the limiting amplitude ($\dot{\theta}_{mL}$ in figure 3) was changed. The three curves correspond to values of $\frac{\theta_1}{\dot{\theta}_{mL}} = 1, 2.0$ and 10.0. It is seen that as the extent of limiting increases the response becomes more and more sluggish corresponding to more damping and less speed of response.

Fig 7. Experimental step responses of the system in figure 3(a) obtained with the simulator for $\theta_{mL}/\theta_1 = 1.0$ (Top), $= 2.0$ (middle) and, $= 10.0$ (bottom).

(b) *Large amplifier gain* :

In this case the system has absolute instability and produces sustained oscillations. In the absence of any velocity limiting, amplitude and frequency of these oscillations are determined by the value of the amplifier gain K and the backlash width. We shall examine the effects of simple velocity limiting on the amplitude and frequency of these oscillations. For this purpose it is most expedient to make use of the gain-phase shift plots. Eqn. (7) is the starting relation. We plot G as a family of amplitude dependent loci on the gain-phase shift diagram. The gain is expressed in *db* and phase shift in degrees. Plot of $-G_B^{-1}$ as an amplitude locus is also superposed on the same diagram. These plots are shown in figure 8 for an assumed value of $T = 1$ and $K = 10$ for the system shown in figure 5(a).

As is evident from Eqn. (7) sustained oscillations of amplitude θ_{ma} (say) are possible only if for this value of signal amplitude $G = -G_B^{-1}$. This condition, if satisfied by the system, can be easily detected from the plots of figure 8. Thus in figure 8 some of the G-loci intersect with the $-G_B^{-1}$ locus. Sustained oscillations will be produced only if and when at the points of intersection the amplitude marked on one of these G-loci is equal to that marked on $-G_B^{-1}$

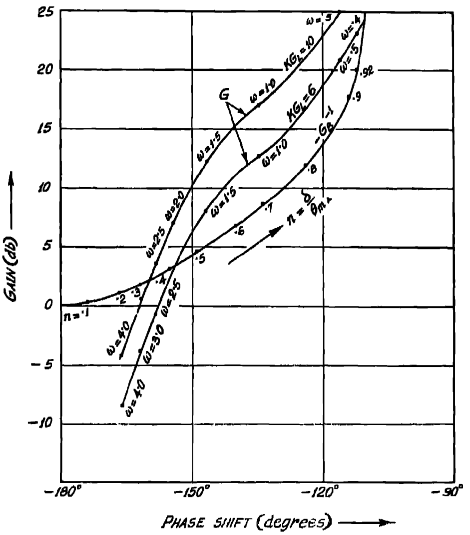

Fig. 8. Gain-phase shift plots of G and G_B^{-1} for the system in figure 3(a) with $K = 10$ and $T = 1$ sec.

locus This is then the amplitude of the resultant sustained oscillations. The frequency of these oscillations is equal to the frequency marked on the relevant G-locus at the point of intersection.

Figure 8 predicts two sets of possible oscillations. It can be shown (Johnson, 1952), however, that oscillations of lower amplitude and frequency correspond to an unstable limit-cycle and no sustained oscillations at this amplitude and frequency are possible.

To see the effect of simple velocity limiting on the amplitude and frequency of oscillations let us assume first that there is no limiting. The amplitude dependence of G-loci then vanishes and all the G-loci converge to a single frequency-locus. Sustained oscillations are produced whenever this locus intersects with the $-G_B^{-1}$ locus. The amplitude of oscillations is that marked on the $-G_B^{-1}$ locus at the point of intersection. Thus in absence of simple velocity limiting the amplitude of oscillation is solely determined by the backlash characteristic. Presence of simple velocity limiting transforms plots of G into a family of amplitude dependent loci and the amplitude of any sustained oscillations are now determined by both the limiter and the backlash characteristics. Suppose the limiting amplitude is set to a certain value and oscillation of some amplitude and frequency is present. If the limiting amplitude is reduced the effective system gain also reduces and from figure 6 it is evident that equilibrium oscillations are now possible only at lower amplitudes. If the limiting amplitude is further reduced oscillation amplitudes also fall but oscillation never ceases, for in absence of oscillation the limiter does not reduce the gain. Thus oscillation amplitudes can be controlled to some extent by imposing simple velocity limiting but can not be stopped altogether.

4. MODIFIED VELOCITY LIMITING

The arrangement is shown in figure 3(b). The corresponding quasilinearized model is shown in figure 5(b). The effects of the limiter in this case also can be studied under the two conditions, viz., small amplifier gain and large amplifier gain. In both the cases similar conclusions as in Sec. 3(a) and 3(b) are arrived at. In the second case, however, an additional remarkable result is obtained.

Thus in the set up shown in figure 3(b) if a step signal is applied at the input the limiter characteristic becomes unsymmetrical as indicated in figure 11 This can be easily verified if we compute the transfer function between the points θ_1 and B which shows a finite d.c. gain. Thus any step of amplitude θ_1 applied at the input will also be present at the input of the limiter (assuming the d.c. gain between θ_1 and B to be unity which is true for $\theta_1 \leqslant L$, L being the limiting amplitude, as shown in figure 11). This will cause the reference lines of the limiter characteristic to be shifted from the position shown by dotted lines to that shown by solid lines. As a result any sustained oscillations that may be present in the system will be subjected to this unsymmetrical characteristics.

Relation (7) is still applicable but now G is dependent not only upon oscillation amplitude but also on the magnitude of any step signal applied at the input. The describing function (G_L') of the unsymmetrical limiter characteristic is derived in Appendix. This is plotted in figure 12 with the amplitude of step input as a parameter. It is seen that for a fixed value of assumed sinusoidal signal amplitude the value of the describing function decreases as the magnitude of the step input is increased.

The effect of input steps on the amplitude and frequency of sustained oscillation in the system of figure 3(b) can be determined by redrawing the curves of figure 8 for a number of assumed input step amplitudes. Thus if we first assume zero input step, we have the curves of figure 8 and the amplitude and frequency of oscillations are found out as outlined in Sec. 3(*b*). If the step input is increased to a finite value, the describing function gain for the oscillations of amplitude determined above will be reduced and the G-locus for this amplitude shifts towards right in figure 8. Thus new equilibrium oscillations will occur at lower amplitude and frequency.

Results obtained with the help of the simulator are presented in fiugure 9 in the form of graphs. For convenience, results obtained by applying the describing function method as outlined in the foregoing paragraphs, are also shown in the figure.

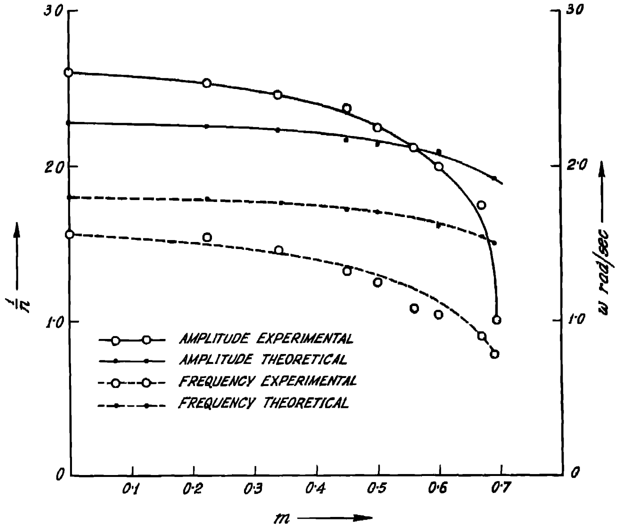

Fig 9 Plots of dimensionless amplitude (θ_{ma}/δ) and frequency (rad./sec.) of sustained oscillations *vs* the dimensionless d.c. input (θ_1/L) for the system in figure 3(b), obtained with the help of the simulator. L is the limiting amplitude in absence of d.c. input.

It is seen that the discrepancy between experimental and theoretical results changes sign and then increases as the amplitude of the input step increases. The increased discrepancies in the results for larger input steps are to be expected, for with increase of the input step amplitude the limiter characteristic becomes more unsymmetrical and the percentage of harmonics produced goes on increasing. Further, oscillation amplitude falls off more rapidly with input step amplitude than indicated by the describing function analysis.

5. ON THE POSSIBILITY OF REALISING THE EFFECTS OF MODIFIED VELOCITY LIMITING

In practice the most common method of limiting the velocity of a servo motor is to place a limiter ahead of it. This has the effect of limiting $\dot{\theta}_m$ and corresponds in effect to the case represented in figure 3(a). It has been shown that a desirable effect from the point of view of system stability can be achieved by incorporating a somewhat different scheme for limiting the motor speed. This case has been referred to as modified velocity limiting and is schematically represented in figure 3(b).

It will be clear from Sec. 4 that the desirable effect of the modified velocity limiting stems from the fact that in this case any d.c. input to the system makes the nonlinearity asymmetrical. The same effect is produced if the nonlinearity is placed in the feedback path as shown in figure 10. It is then possible to achieve the effects of modified velocity limiting by the system of figure 10.

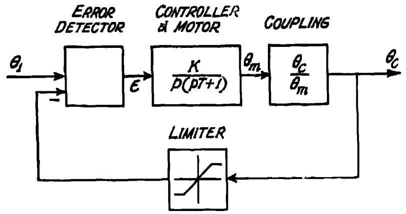

Fig 10 Alternative configuration for realising the effects of modified velocity limiting system stability.

It is to be noted that in figure 10 the two nonlinearities are no longer separated by low-pass units and the describing function method does not apply. That is why in the foregoing analyses the system of figure 3(b) was chosen instead of that of figure 10.

6. CONCLUSION

It is evident from the above studies that effect of the presence of simple velocity limiting in a servomechanism containing backlash is to improve the relative stability of the system for larger signals. It is thus possible to reduce, for large signals, the derogatory effects of backlash by inclusion of simple velocity limiting. This has however the severe drawback, viz., that this improvement in damping constant as a result of velocity limiting is associated with a comparatively large reduction in speed of response.

It has also been found that simple velocity limiting has no stabilising effect on a system caused to have absolute instability in the sense that it can not stop oscillations but can be used to control the amplitude of oscillations present. A remarkable result is obtained in this case by introduction of modified velocity limiting. It is seen that amplitude of oscillation is in this case dependent on magnitude of any d.c input signal, and that oscillation ceases entirely as the d.c. input signal amplitude increases above certain value. This result is of some importance because in practice inputs to servo systems are in most cases random time functions and signals of certain amplitudes are more likely to occur than others. Since the oscillations due to backlash effect are essentially of small amplitudes superimposed on the comparatively large displacements due to signals it is evident that it is possible to adjust the limiter characteristic in the case of modified velocity limiting so as to make any oscillation negligible.

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APPENDIX

Describing Function of the Unsymmetrical Limiter Characteristics

The limiter characteristic assumed is shown in figure 11. If L is the limiting amplitude for the symmetrical case as shown in dotted lines in figure 11 and θ_1 is the value of the positive d.c. input to the limiter, the new limiting values are $L-\theta_1$ and $-(L+\theta_1)$ on the positive and negative sides respectively.

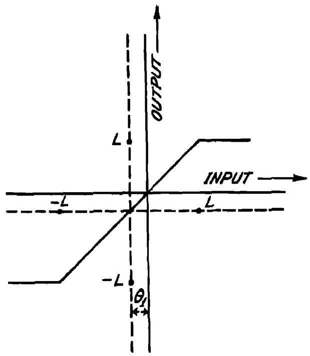

Fig. 11. Unsymmetrical characteristic of the limiter in figure 3(b) resulting from d.c. input θ_1. Dotted lines show the positions of the reference lines in absence of any d.c. input.

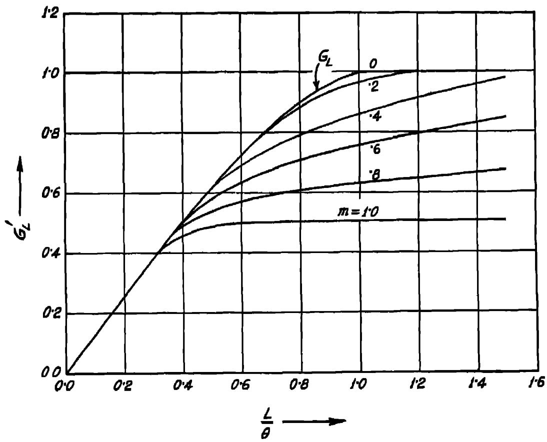

Fig. 12. Describing functions of the unsymmetrical limiter of figure 10. $m = \theta_1/L$, the non-dimensional d.c. input, θ_a is the amplitude of assumed sine wave input to the limiter.

As usual we assume the input to be a sinusoidal function of the form

$$\theta_{in} = \theta_a \sin \omega t.$$

The output of the limiter is then defined by

$$\theta_{out} = \theta_a \sin \omega t \; ; \; 0 \leqslant \omega t \leqslant \sin^{-1} \frac{L-\theta_1}{\theta_a}$$

$$= L-\theta_1 \quad ; \; \sin^{-1} \frac{L-\theta_1}{\theta_a} \leqslant \omega t \leqslant \pi - \sin^{-1} \frac{L-\theta_1}{\theta_a}$$

$$= \theta_a \sin \omega t \; ; \; \pi - \sin^{-1} \frac{L-\theta_1}{\theta_a} \leqslant \omega t \leqslant \pi.$$

for the positive half period of θ_{in} and

$$\theta_{out} = \theta_a \sin \omega t \; ; \; \pi \leqslant \omega t \leqslant \pi + \sin^{-1} \frac{-(L+\theta_1)}{\theta_a}$$

$$= -(L+\theta_1) \; ; \; \pi + \sin^{-1} \frac{-(L+\theta_1)}{\theta_a} \leqslant \omega t \leqslant 2\pi - \sin^{-1} \frac{-(L+\theta_1)}{\theta_a}$$

$$= \theta_a \sin \omega t \; ; \; 2\pi - \sin^{-1} \frac{-(L+\theta_1)}{\theta_a} \leqslant \omega t \leqslant 2\pi,$$

for the negative half period of θ_{in}.

By applying Fourier's expansion to θ_{out} we can find the value of the fundamental components for the two halfs. This gives

$$[\theta_{out}]_f = \frac{\theta_a}{\pi} \left[\sin^{-1} \frac{L-\theta_1}{\theta_a} + \frac{L-\theta_1}{\theta_a} \cos \sin^{-1} \frac{L-\theta_1}{\theta_a} \right] \sin \omega t$$

$$+ \frac{\theta_a}{\pi} \left[\sin^{-1} \frac{L+\theta_1}{\theta_a} + \frac{L+\theta_1}{\theta_a} \cos \sin^{-1} \frac{L+\theta_1}{\theta_a} \right] \sin \omega t$$

the describing function

$$G'_L = \frac{1}{\pi} \left\{ \left[\sin^{-1} \frac{L-\theta_1}{\theta_a} + \frac{L-\theta_1}{\theta_a} \cos \sin^{-1} \frac{L-\theta_1}{\theta_a} \right] \right.$$

$$\left. + \sin^{-1} \frac{L+\theta_1}{\theta_a} + \frac{L+\theta_1}{\theta_a} \cos \sin^{-1} \frac{L+\theta_1}{\theta_a} \right] \right\}$$

$$= \frac{1}{\pi} \left\{ \left[\sin^{-1} \frac{L}{\theta_a} (1-m) + \frac{L}{\theta_a} (1-m) \cos \sin^{-1} \frac{L}{\theta_a} (1-m) \right] \right. -$$

$$\left. + \left[\sin^{-1} \frac{L}{\theta_a} (1+m) + \frac{L}{\theta_a} (1+m) \cos \sin^{-1} \frac{L}{\theta_a} (1+m) \right] \right\},$$

where

$$m = \theta_1/L.$$

This is plotted in figure 12 as a function of (L/θ_a) with m as a parameter.